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ORIGINAL PAPER

Properties of the two-dimensional exponential integral functions

Yelda Aygar · Elgiz Bairamov

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Abstract In this paper uniform convergences of the two-dimensional exponential integral functions are investigated. We also study the continuity, integrability and asymptotic behaviour of these functions.

Keywords Multidimensional radiative transfer · Isotropic scattering · Two-dimensional exponential integral functions · Uniform convergence

1 Introduction

Let us consider the functions

$$\mathcal{E}_1(\tau,\beta)=\int\limits_1^\infty \left(t^2+\beta^2\right)^{-\frac{1}{2}}\exp\left[-\tau\left(t^2+\beta^2\right)^{\frac{1}{2}}\right]dt, \tag{1.1}$$

$$\mathcal{E}_2(\tau,\beta)=\int\limits_1^\infty t^{-2}\exp\left[-\tau\left(t^2+\beta^2\right)^{\frac{1}{2}}\right]dt, \tag{1.2}$$

$$\mathcal{E}_3(\tau,\beta)=\tau\int\limits_1^\infty \mathcal{E}_2\left(\tau t,\frac{\beta}{t}\right)dt, \tag{1.3}$$

Y. Aygar (✉) · E. Bairamov
Department of Mathematics, Ankara University, 06100 Tandoğan, Ankara, Turkey
e-mail: yaygar@science.ankara.edu.tr

E. Bairamov
e-mail: bairamov@science.ankara.edu.tr

where $(\tau, \beta) \in \Omega := \left\{(\tau, \beta) \in \mathbb{R}^2,\ \tau \in [0, \infty),\ \beta \in (-\infty, \infty)\right\}$. These function play an important role in the study of radiative transfer in a two-dimensional planar medium [1–9]. In particular, the function $\mathcal{E}_1(\tau, \beta)$ is the kernel of the Fredholm integral equations describing isotropic scattering and radiative equilibrium [1–9]. In studying the absorption of solar radiation by the earth's atmosphere, Chapman studied the function

$$\varphi(x, u) = xue^x \int_\alpha^\infty \cosh t \exp[-xu \cosh t]\, dt$$

where $\alpha = \arccos \frac{1}{u}$ [10]. Breig and Crosbie shows that, Chapman's function is related to $\mathcal{E}_1(\tau, \beta)$ as follows [11]:

$$-\tau \frac{\partial \mathcal{E}_1(\tau, \beta)}{\partial \tau} = \exp\left[-\tau \left(t^2 + \beta^2\right)^{\frac{1}{2}}\right] \varphi\left[\tau \left(1 + \beta^2\right)^{\frac{1}{2}}, \beta \left(1 + \beta^2\right)^{-\frac{1}{2}}\right].$$

Series expansions and numerical computation of the values of the functions (1.1)–(1.3) are given in [11]. Several procedures for evaluating the functions $\mathcal{E}_n(\tau, \beta)$, $n = 1, 2, 3$ can be found in [12–15].

The functions $\mathcal{E}_n(\tau, \beta)$, $n = 1, 2, 3$ are two-dimensional analogs of the exponential integral functions [16]

$$E_n(\tau) = \int_1^\infty t^{-n} \exp(-\tau t)\, dt, \tag{1.4}$$

$n = 1, 2, 3$. It is clear that, $\mathcal{E}_n(\tau, 0) = E_n(\tau)$, $n = 1, 2, 3$. The exponential integral function (1.4) plays an important role in various fields of theoretical physics, quantum chemistry, theory of fluid flow and theory of transport process [17–26].

Many properties of the functions $\mathcal{E}_n(\tau, \beta)$ $n = 1, 2, 3$ depend on the uniform convergence of the improper integrals (1.1)–(1.3). In this paper, we study the uniform convergence of the functions $\mathcal{E}_n(\tau, \beta)$ $n = 1, 2, 3$. We also investigate the continuity, integrability and asymptotic behaviour of (1.1)–(1.3).

2 Uniform convergence

Let us consider the improper integreal

$$\int_1^\infty f(\tau, \beta, t)\, dt, \tag{2.1}$$

where $(\tau, \beta) \in D \subset \mathbb{R}^2$.

Definition 2.1 [27] Let for every $(\tau, \beta) \in D$ the integral (2.1) exists. If for any $\varepsilon > 0$ there exists a number $a_\varepsilon \geq 1$ not depend on (τ, β) such that for all $A > a_\varepsilon$, the inequality

$$\left| \int_A^\infty f(\tau, \beta, t)\, dt \right| < \varepsilon$$

holds for all $(\tau, \beta) \in D$, then the integral (2.1) is said to be uniformly convergent with respect to (τ, β) on D.

Theorem 2.1 [27] *If there exists a non-negative function $\varphi(t)$ only which is integrable over $[1, \infty)$ and is such that for all $(\tau, \beta) \in D$, $t \in [1, \infty)$*

$$|f(\tau, \beta, t)| \leq \varphi(t)$$

then the integral (2.1) convergences uniformly with respect to (τ, β) on D.

Now let us consider the improper integral

$$\int_1^\infty g(\tau, \beta, t)\, \psi(\tau, \beta, t)\, dt. \tag{2.2}$$

Theorem 2.2 [28] *If the improper integral*

$$\int_1^\infty g(\tau, \beta, t)\, dt$$

convergences uniformly with respect to (τ, β) on the domain D and the function $\psi(\tau, \beta, t)$ is uniformly bounded, that is

$$|\psi(\tau, \beta, t)| \leq M, \ \ (\tau, \beta) \in D, \ t \in [1, \infty),$$

where M is a constant not depending on (τ, β, t), then the integral (2.2) convergences uniformly with respect to (τ, β) on D.

For all $\varepsilon > 0$ we define the domain

$$D(\varepsilon) = \{(\tau, \beta) \in \Omega, \ \tau \in [\varepsilon, \infty), \ \beta \in (-\infty, \infty)\},$$

where

$$\Omega = \left\{(\tau, \beta) \in \mathbb{R}^2, \ \tau \in [0, \infty), \ \beta \in (-\infty, \infty)\right\}.$$

Theorem 2.3 i) *The two-dimensional exponential integral function* $\mathcal{E}_1(\tau, \beta)$ *is uniformly convergent with respect to* (τ, β) *on* $D(\varepsilon)$.
ii) *The function* $\mathcal{E}_1(\tau, \beta)$ *is nonuniformly convergent with respect to* (τ, β) *on* Ω.

Proof i) If we define

$$g(\tau, \beta, t) = \exp\left[-\tau\left(t^2+\beta^2\right)^{\frac{1}{2}}\right],\ \psi(\tau, \beta, t) = \left(t^2+\beta^2\right)^{-\frac{1}{2}},$$

then

$$\mathcal{E}_1(\tau, \beta) = \int_1^\infty g(\tau, \beta, t)\, \psi(\tau, \beta, t)\, dt, \tag{2.3}$$

by (1.1). We have

$$g(\tau, \beta, t) \leq \exp(-\varepsilon t), (\tau, \beta) \in D(\varepsilon), t \in [1, \infty).$$

So for all $\varepsilon > 0$

$$\int_1^\infty \exp(-\varepsilon t)\, dt = \varepsilon^{-1} \exp(-\varepsilon) < \infty,$$

then by Theorem 2.1 the integral

$$\int_1^\infty g(\tau, \beta, t)\, dt \tag{2.4}$$

is uniformly convergent with respect to (τ, β) on $D(\varepsilon)$. We easily obtain that

$$|\psi(\tau, \beta, t)| \leq 1,\ \ (\tau, \beta) \in D(\varepsilon),\ t \in [1, \infty), \tag{2.5}$$

i.e., the function $\psi(\tau, \beta, t)$ is uniformly bounded. It follows (2.4), (2.5) and Theorem 2.2 that the integral (2.3) is uniformly convergent with respect to (τ, β) on $D(\varepsilon)$.

ii) For all $(\tau, \beta) \in \Omega$ we have

$$\mathcal{E}_1(\tau, \beta) = \int_{\tau(1+\beta^2)^{\frac{1}{2}}}^{\infty} \left(u^2 - \tau^2\beta^2\right)^{-\frac{1}{2}} \exp(-u)\, du \geq \int_{\tau(1+\beta^2)^{\frac{1}{2}}}^{\infty} u^{-1} \exp(-u)\, du \to \infty \quad \text{for } \tau \to 0.$$

Therefore the function $\mathcal{E}_1(\tau, \beta)$ is nonuniformly convergent with respect to (τ, β) on Ω. □

Theorem 2.4 *The integrals* (1.2) *and* (1.3) *are uniformly convergent with respect to* (τ, β) *on* Ω *and* $D(\varepsilon)$*, respectively.*

Proof Suppose that $(\tau, \beta) \in \Omega$. It can easily be shown that

$$t^{-2} \exp\left[-\tau\left(t^2 + \beta^2\right)^{\frac{1}{2}}\right] \leq t^{-2}.$$

Since

$$\int_1^{\infty} t^{-2} dt = 1,$$

we see that by Theorem 2.1 the integral (1.2) is uniformly convergent with respect to (τ, β) on Ω.

Now consider the function

$$\begin{aligned}\mathcal{E}_3(\tau, \beta) &= \tau \int_1^{\infty} \mathcal{E}_2\left(\tau t, \frac{\beta}{t}\right) dt \\ &= \tau \int_1^{\infty}\int_1^{\infty} s^{-2} \exp\left[-\tau\left(t^2 s^2 + \beta^2\right)^{\frac{1}{2}}\right] ds dt \\ &= \tau J(\tau, \beta) \end{aligned} \tag{2.6}$$

where

$$J(\tau, \beta) = \int_1^{\infty}\int_1^{\infty} s^{-2} \exp\left[-\tau\left(t^2 s^2 + \beta^2\right)^{\frac{1}{2}}\right] ds dt \tag{2.7}$$

For all $(\tau, \beta) \in D(\varepsilon)$ we get

$$s^{-2} \exp\left[-\tau\left(t^2 s^2+\beta^2\right)^{\frac{1}{2}}\right] \le s^{-2} \exp(-\varepsilon t s). \tag{2.8}$$

Since

$$\begin{aligned}\int_1^\infty \int_1^\infty s^{-2} \exp(-\varepsilon t s)\, ds dt &= \int_1^\infty s^{-2}\left\{\int_1^\infty \exp(-\varepsilon t s)\, dt\right\} ds \\ &= \int_1^\infty s^{-3}\varepsilon^{-1} \exp(-\varepsilon s) \le \varepsilon^{-1} \exp(-\varepsilon) \int_1^\infty s^{-3} ds \\ &= 2^{-1}\varepsilon^{-1} \exp(-\varepsilon),\end{aligned}$$

we see that by (2.6)–(2.8) and Theorem 2.1 that the integral (1.3) is uniformly convergent with respect to (τ, β) on $D(\varepsilon)$.

Let $(\tau, \beta) \in \Omega$. Using (2.7) we get

$$\begin{aligned}J(\tau, \beta) &= \int_1^\infty \int_1^\infty s^{-2} \exp\left[-\tau\left(t^2 s^2+\beta^2\right)^{\frac{1}{2}}\right] ds dt \\ &= \int_1^\infty s^{-2}\left\{\int_{(s^2+\beta^2)^{\frac{1}{2}}}^\infty s^{-1}\left(u^2-\beta^2\right)^{-\frac{1}{2}} u \exp(-\tau u)\, du\right\} ds \\ &\ge \int_1^\infty s^{-3}\left\{\int_{(s^2+\beta^2)^{\frac{1}{2}}}^\infty \exp(-\tau u)\, du\right\} ds \\ &= \int_1^\infty s^{-3}\tau^{-1} \exp\left[-\tau\left(s^2+\beta^2\right)^{\frac{1}{2}}\right] ds \\ &= \tau^{-1} \int_1^\infty s^{-3} \exp\left[-\tau\left(s^2+\beta^2\right)^{\frac{1}{2}}\right] ds \to \infty \quad \text{for } \tau \to 0\end{aligned}$$

so we obtain the following.

Theorem 2.5 *The integral* (1.3) *is non-uniformly convergent with respect to* (τ, β) *on* Ω.

3 Properties of the function $\mathcal{E}_n(\tau,\beta)$

We will use the following well known theorems.

Theorem 3.1 [27] *If the function* $f(\tau,\beta,t)$ *is continuous on*

$$D\times[1,\infty):=\{(\tau,\beta)\in D,\ t\in[1,\infty)\}$$

and if the integral (2.1) *is uniformly convergent with respect to* (τ,β) *on D, then the function*

$$F(\tau,\beta)=\int_1^\infty f(\tau,\beta,t)\,dt$$

is the continuous function with respect to (τ,β) *on D and for all* $(\tau_0,\beta_0)\in D$

$$\lim_{\substack{\tau\to\tau_0\\ \beta\to\beta_0}}\int_1^\infty f(\tau,\beta,t)\,dt=\int_1^\infty \lim_{\substack{\tau\to\tau_0\\ \beta\to\beta_0}} f(\tau,\beta,t)\,dt.$$

Theorem 3.2 [27] *If the conditions of Theorem 3.1 hold and D is a measurable domain in* $\mathbb{R}^2$*, then the function* $F(\tau,\beta)$ *can be integrated with respect to* (τ,β) *on D under the integral sign, that is*

$$\iint_D F(\tau,\beta)\,d\tau d\beta=\int_1^\infty\left[\iint_D f(\tau,\beta,t)\,d\tau d\beta\right]dt.$$

Theorem 3.3 i) *The functions* $\mathcal{E}_1(\tau,\beta)$ *and* $\mathcal{E}_3(\tau,\beta)$ *are continuous with respect to* (τ,β) *on the domain* $D(\varepsilon)$.
ii) *The function* $\mathcal{E}_2(\tau,\beta)$ *is continuous with respect to* (τ,β) *on* Ω.

Proof i) Let (τ_0,β_0) is any point of $D(\varepsilon)$. Using Theorems 2.3 and 3.1 we get

$$\begin{aligned}\lim_{\substack{\tau\to\tau_0\\ \beta\to\beta_0}}\mathcal{E}_1(\tau,\beta)&=\lim_{\substack{\tau\to\tau_0\\ \beta\to\beta_0}}\int_1^\infty\left(t^2+\beta^2\right)^{-\frac{1}{2}}\exp\left[-\tau\left(t^2+\beta^2\right)^{\frac{1}{2}}\right]dt\\&=\int_1^\infty\lim_{\substack{\tau\to\tau_0\\ \beta\to\beta_0}}\left\{\left(t^2+\beta^2\right)^{-\frac{1}{2}}\exp\left[-\tau\left(t^2+\beta^2\right)^{\frac{1}{2}}\right]\right\}dt\\&=\int_1^\infty\left(t^2+\beta_0^2\right)^{-\frac{1}{2}}\exp\left[-\tau_0\left(t^2+\beta_0^2\right)^{\frac{1}{2}}\right]dt=\mathcal{E}_1(\tau_0,\beta_0).\end{aligned}$$

Therefore the function $\mathcal{E}_1(\tau,\beta)$ is continuous on $D(\varepsilon)$. In a similar way we can prove the continuity of the functions $\mathcal{E}_2(\tau,\beta)$ and $\mathcal{E}_3(\tau,\beta)$ with respect to (τ,β) on Ω and $D(\varepsilon)$, respectively. □

Theorem 3.4 i) *$\mathcal{E}_1(\tau,\beta)$ satisfy the following asymptotic equations:*

$$\mathcal{E}_1(\tau,\beta) = o(1), \ (\tau,\beta)\in D(\varepsilon), \ \tau\to\infty. \tag{3.1}$$

$$\mathcal{E}_1(\tau,\beta) = o(1), \ (\tau,\beta)\in D(\varepsilon), \ \beta\to\pm\infty. \tag{3.2}$$

$$\mathcal{E}_1(\tau,\beta) = E_1(\tau) + o(1), \ (\tau,\beta)\in D(\varepsilon), \ \beta\to 0. \tag{3.3}$$

ii) $\mathcal{E}_1(\tau,\beta)\in L^1[D(\varepsilon)]$,
where

$$L^1[D(\varepsilon)] := \left\{ F = F(\tau,\beta), \ \iint\limits_{D(\varepsilon)} |F(\tau,\beta)|\, d\tau d\beta < \infty \right\}.$$

Proof i) Using Theorems 2.3 and 3.1, we obtain that, for all $(\tau,\beta)\in D(\varepsilon)$

$$\lim_{\tau\to\infty}\mathcal{E}_1(\tau,\beta) = \lim_{\tau\to\infty}\int_1^\infty \left(t^2+\beta^2\right)^{-\frac{1}{2}} \exp\left[-\tau\left(t^2+\beta^2\right)^{\frac{1}{2}}\right] dt$$
$$= \int_1^\infty \left(t^2+\beta^2\right)^{-\frac{1}{2}} \lim_{\tau\to\infty}\exp\left[-\tau\left(t^2+\beta^2\right)^{\frac{1}{2}}\right] dt = 0,$$

i.e., (3.1) holds. In a similar way we show (3.2).
Let us prove (3.3).

$$\begin{aligned}\mathcal{E}_1(\tau,\beta) - E_1(\tau) &= \int_1^\infty \left(t^2+\beta^2\right)^{-\frac{1}{2}} \exp\left[-\tau\left(t^2+\beta^2\right)^{\frac{1}{2}}\right] dt \\ &\quad - \int_1^\infty t^{-1}\exp(-\tau t)\, dt \\ &= \int_1^\infty \left[\left(t^2+\beta^2\right)^{-\frac{1}{2}} - t^{-1}\right] \exp\left[-\tau\left(t^2+\beta^2\right)^{\frac{1}{2}}\right] dt \\ &\quad + \int_1^\infty t^{-1}\left\{\exp\left[-\tau\left(t^2+\beta^2\right)^{\frac{1}{2}}\right] - \exp(-\tau t)\right\} dt \\ &= I_1(\tau,\beta) + I_2(\tau,\beta),\end{aligned} \tag{3.4}$$

where

$$I_1(\tau,\beta)=\int_1^{\infty}\left[\left(t^2+\beta^2\right)^{-\frac{1}{2}}-t^{-1}\right]\exp\left[-\tau\left(t^2+\beta^2\right)^{\frac{1}{2}}\right]dt=\int_1^{\infty}g_1(\tau,\beta,t)\,dt \tag{3.5}$$

$$I_2(\tau,\beta)=\int_1^{\infty}t^{-1}\left\{\exp\left[-\tau\left(t^2+\beta^2\right)^{\frac{1}{2}}\right]-\exp(-\tau t)\right\}dt=\int_1^{\infty}g_2(\tau,\beta,t)\,dt. \tag{3.6}$$

Since for all $(\tau,\beta)\in D(\varepsilon)$

$$\begin{aligned}|g_1(\tau,\beta,t)|&=\left|\left(t^2+\beta^2\right)^{-\frac{1}{2}}-t^{-1}\right|\exp\left[-\tau\left(t^2+\beta^2\right)^{\frac{1}{2}}\right]\\&\le\left[\left(t^2+\beta^2\right)^{-\frac{1}{2}}+t^{-1}\right]\exp(-\varepsilon t)\le 2\exp(-\varepsilon t),\end{aligned}$$

and

$$|g_2(\tau,\beta,t)|\le 2\exp(-\varepsilon t),$$

hold. Therefore by Theorem 2.1 the integrals (3.5) and (3.6) are uniformly convergent with respect to (τ,β) on $D(\varepsilon)$. From (3.4) and Theorem 3.1 we have

$$\begin{aligned}\lim_{\beta\to 0}\left[\mathcal{E}_1(\tau,\beta)-E_1(\tau)\right]&=\lim_{\beta\to 0}\left[I_1(\tau,\beta)+I_2(\tau,\beta)\right]\\&=\lim_{\beta\to 0}\int_1^{\infty}\left[\left(t^2+\beta^2\right)^{-\frac{1}{2}}-t^{-1}\right]\exp\left[-\tau\left(t^2+\beta^2\right)^{\frac{1}{2}}\right]dt\\&\quad+\lim_{\beta\to 0}\int_1^{\infty}t^{-1}\left\{\exp\left[-\tau\left(t^2+\beta^2\right)^{\frac{1}{2}}\right]-\exp(-\tau t)\right\}dt\\&=0\end{aligned}$$

i.e., (3.3) holds.

ii) It follows from Theorems 2.3 and 3.2 that

$$\begin{aligned}\int\int_{D(\varepsilon)}\mathcal{E}_1(\tau,\beta)\,d\tau d\beta&=\int\int_{D(\varepsilon)}\left\{\int_1^{\infty}\left(t^2+\beta^2\right)^{-\frac{1}{2}}\exp\left[-\tau\left(t^2+\beta^2\right)^{\frac{1}{2}}\right]dt\right\}d\tau d\beta\\&=\int_1^{\infty}\left\{\int\int_{D(\varepsilon)}\left(t^2+\beta^2\right)^{-\frac{1}{2}}\exp\left[-\tau\left(t^2+\beta^2\right)^{\frac{1}{2}}\right]d\tau d\beta\right\}dt\end{aligned}$$

$$= \int_1^{\infty} \left\{ \int_{-\infty}^{\infty} \int_{\varepsilon}^{\infty} \left(t^2+\beta^2\right)^{-\frac{1}{2}} \exp\left[-\tau\left(t^2+\beta^2\right)^{\frac{1}{2}}\right] d\tau d\beta \right\} dt$$
$$= \int_1^{\infty} \int_{-\infty}^{\infty} \left(t^2+\beta^2\right)^{-\frac{1}{2}} \left\{ \int_{\varepsilon}^{\infty} \exp\left[-\tau\left(t^2+\beta^2\right)^{\frac{1}{2}}\right] d\tau \right\} d\beta dt$$
$$= \int_1^{\infty} \int_{-\infty}^{\infty} \left(t^2+\beta^2\right)^{-1} \exp\left[-\varepsilon\left(t^2+\beta^2\right)^{\frac{1}{2}}\right] d\beta dt$$
$$= 2\int_1^{\infty} \int_0^{\infty} \left(t^2+\beta^2\right)^{-1} \exp\left[-\varepsilon\left(t^2+\beta^2\right)^{\frac{1}{2}}\right] d\beta dt$$
$$\leq 2\int_1^{\infty} t^{-2} dt \int_0^{\infty} \exp\left(-\varepsilon\beta\right) d\beta = \frac{2}{\varepsilon} < \infty.$$

So we have $\mathcal{E}_1(\tau,\beta) \in L^1[D(\varepsilon)]$. □

Similar to Theorem 3.4 we get the following:

Theorem 3.5 *The functions $\mathcal{E}_2(\tau,\beta)$ and $\mathcal{E}_3(\tau,\beta)$ satisfy*

$$\mathcal{E}_2(\tau,\beta) = o(1),\ \ (\tau,\beta) \in \Omega,\ \tau \to \infty$$
$$\mathcal{E}_2(\tau,\beta) = o(1),\ (\tau,\beta) \in \Omega,\ \beta \to \pm\infty$$
$$\mathcal{E}_2(\tau,\beta) = E_2(\tau) + o(1),\ \ (\tau,\beta) \in \Omega, \beta \to 0$$
$$\mathcal{E}_2(\tau,\beta) = 1 + o(1),\ \ (\tau,\beta) \in \Omega,\ \tau \to 0$$

$$\mathcal{E}_3(\tau,\beta) = o(1),\ \ (\tau,\beta) \in D(\varepsilon),\ \tau \to \infty$$
$$\mathcal{E}_3(\tau,\beta) = o(1),\ \ (\tau,\beta) \in D(\varepsilon),\ \beta \to \pm\infty$$
$$\mathcal{E}_3(\tau,\beta) = E_3(\tau) + o(1),\ \ (\tau,\beta) \in D(\varepsilon),\ \beta \to 0.$$

4 Numerical results

On the basis of the uniform convergence of the function $\mathcal{E}_n(\tau,\beta)$, obtained in this paper we constructed a program in the Maple Software®. The computations were performed for wide values of the parameters τ and β. As can be seen from Table 1, the calculation results of the functions $\mathcal{E}_n(\tau,\beta)$, $(n = 1, 2)$ show good rate of convergence in the range of parameters $\tau \epsilon (0, 1]$ and $\beta \geq 10$.

Table 1 Using the Maple Software® we give the following values of the function $\mathcal{E}_n(\tau, \beta)$

n	β	$\tau = 10^{-3}$	$\tau = 10^{-1}$	$\tau = 1$
1	10	*A* 4.62240566	*A* 3.8435848E-01	*A* 1.3321817E-05
		B 4.6224	*B* 3.8436-01	*B* 1.332E-05
1	20	*A* 3.97946820	*A* 1.0713555E-01	*A* 4.7196045E-10
		B 3.9795	*B* 1.0714E-01	*B* 4.7196E-10
1	50	*A* 3.09521077	*A* 3.5563933E-03	*A* 3.0257252E-23
		B 3.0952	*B* 3.5564E-03	*B* 3.026E-23
2	10	*A* 9.8537805E-01	*A* 3.2761334E-01	*A* 2.9869304E-05
		B 9.8538E-01	*B* 3.2761E-01	*B* 2.9869E-05
2	20	*A* 9.7619472E-01	*A* 1.2428017E-01	*A* 1.5383345E-09
		B 9.7619E-01	*B* 1.2428E-01	*B* 1.5383E-09
2	50	*A* 9.48112470E-01	*A* 6.3756239E-03	*A* 1.6069879E-22
		B 9.4812E-01	*B* 6.3756E-03	*B* 1.607E-22

A our computational result in Table 1
B computational result of ref. [11]

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